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Modified Bernstein Polynomials and Jacobi Polynomials in q -Calculus

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Abstract

We introduce here a generalization of the modified Bernstein polynomials for Jacobi weights using the q -Bernstein basis proposed by G.M. Phillips to generalize classical Bernstein Polynomials. The function is evaluated at points which are in geometric progression in $]0, 1[$. Numerous properties of the modified Bernstein Polynomials are extended to their q -analogues: simultaneous approximation, pointwise convergence even for unbounded functions, shape-preserving property, Voronovskaya theorem, self-adjointness. Some properties of the eigenvectors, which are q -extensions of Jacobi polynomials, are given.

Keywords: q -Bernstein, q -Jacobi, Bernstein-Durrmeyer, totally positive, simultaneous approximation.

AMS subject classification: 41A10, 41A25, 41A36

1 Introduction

G.M.Phillips has proposed a generalization of Bernstein polynomials based on the q -integers (cf. [9]). We introduce here a q -analogue of the operators which are often called Bernstein Durrmeyer polynomials and denoted $M_{n,1}^{\alpha,\beta}$ (cf. [3],[2]).

In all the paper, we shall assume that $q \in]0, 1[$ and, $\alpha, \beta > -1$ (part 5 excepted).

For any integer n , and a function f defined on $]0, 1[$ we set

$$M_{n,q}^{\alpha,\beta} f(x) = \sum_{k=0}^n f_{n,k,q}^{\alpha,\beta} b_{n,k,q}(x) \quad (1)$$

where each $f_{n,k,q}^{\alpha,\beta}$ is a mean of f defined by Jackson integrals. The polynomials $b_{n,k,q}(x)$ are q -analogues of the Bernstein basis polynomials and are defined by $b_{n,k,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k}$, with $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$, (q -binomial coefficient), $k = 0, \dots, n$. They verify $\sum_{k=0}^n b_{n,k,q}(x) = 1$ (cf. [9]).

We follow the definitions and notations of ([7]).

For any real a , $[a]_q = (1 - q^a)/(1 - q)$, $\Gamma_q(a+1) = (1 - q)_q^a/(1 - q)^a$,

(if $a \in \mathbb{N}$ the q -integer $[a]_q$ is $[a]_q = 1 + q + \dots + q^{a-1}$ and $\Gamma_q(a+1) = ([a]_q!)$);

$(1 - x)_q^a = \prod_{j=0}^{\infty} (1 - q^j x) / \prod_{j=0}^{\infty} (1 - q^{j+a} x)$ and consequently $(1 - x)_q^{a+b} = (1 - x)_q^a (1 - q^a x)_q^b$ holds for any b , $(1 - x)_q^m = \prod_{j=0}^{m-1} (1 - q^j x)$ if m is integer.

The notations will be simplified as much as possible, the superscript α, β and the index q when q is fixed, will be suppressed in some proofs.

We introduce the positive bilinear form:

$$\langle f, g \rangle_q^{\alpha,\beta} = q^{(\alpha+1)(\beta+1)} (1 - q) \sum_{i=0}^{\infty} q^i q^{i\alpha} (1 - q^{i+1})^\beta f(q^{i+\beta+1}) g(q^{i+\beta+1}), \quad (2)$$

whenever it is defined. It can be written under the form of two definite q -integrals

$$\begin{aligned}\langle f, g \rangle_q^{\alpha, \beta} &= \int_0^{q^{\beta+1}} t^\alpha (1 - q^{-\beta} t)_q^\beta f(t) g(t) d_q t \\ \text{and } \langle f, g \rangle_q^{\alpha, \beta} &= q^{(\alpha+1)(\beta+1)} \int_0^1 t^\alpha (1 - qt)_q^\beta f(q^{\beta+1} t) g(q^{\beta+1} t) d_q t.\end{aligned}$$

(the definite q -integral of a function f is $\int_0^a f(x) d_q x = a(1-q) \sum_{i=0}^{\infty} q^i f(q^i a)$ (cf. [7]))

Definition 1 We set in formula (1) :

$$f_{n,k,q}^{\alpha, \beta} = \frac{\langle b_{n,k,q}, f \rangle_q^{\alpha, \beta}}{\langle b_{n,k,q}, 1 \rangle_q^{\alpha, \beta}} = \frac{\int_0^1 t^{k+\alpha} (1 - qt)_q^{n-k+\beta} f(q^{\beta+1} t) d_q t}{\int_0^1 t^{k+\alpha} (1 - qt)_q^{n-k+\beta} d_q t}, \quad k = 0, \dots, n, \quad (3)$$

$$\text{to define } M_{n,q}^{\alpha, \beta} f(x) = \sum_{k=0}^n \frac{\langle b_{n,k,q}, f \rangle_q^{\alpha, \beta}}{\langle b_{n,k,q}, 1 \rangle_q^{\alpha, \beta}} b_{n,k,q}(x). \quad (4)$$

We see that the polynomial $M_{n,q}^{\alpha, \beta} f$ is well defined if there exists $\gamma \geq 0$ such that $x^\gamma f(x)$ is bounded on $]0, A]$ for some $A \in]0, 1]$ and $\alpha > \gamma - 1$. Indeed, $x^\alpha f(x)$ is then q -integrable for the weight $w_q^{\alpha, \beta}(x) = x^\alpha (1 - qx)_q^\beta$. We will say, in this case, that f satisfies the condition $C(\alpha)$. Also $\langle f, g \rangle_q^{\alpha, \beta}$ is well defined if the product fg satisfies $C(\alpha)$, particularly if f^2 and g^2 do it.

In many cases, the limit of $M_{n,q}^{\alpha, \beta} f(x)$ when q tends to 1 is :

$$M_{n,1}^{\alpha, \beta} f(x) = \sum_{k=0}^n \left(\int_0^1 t^{k+\alpha} (1-t)^{n-k+\beta} f(t) dt \Big/ \int_0^1 t^{k+\alpha} (1-t)^{n-k+\beta} dt \right) b_{n,k}(x)$$

with $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Numerous properties of the operator $M_{n,1}^{\alpha, \beta}$ will be extended to $M_{n,q}^{\alpha, \beta}$ in this paper.

2 First properties

For any $n \in \mathbb{N}$, the operator $M_{n,q}^{\alpha,\beta}$ has the following properties.

- It is linear, positive and it preserves the constants so it is a contraction

$$\left(\sup_{x \in [0,1]} |M_{n,q}^{\alpha,\beta} f(x)| \leq \sup_{x \in]0,1[} |f(x)| \right).$$

- It is self-adjoint: $\langle M_{n,q}^{\alpha,\beta} f, g \rangle_q^{\alpha,\beta} = \langle f, M_{n,q}^{\alpha,\beta} g \rangle_q^{\alpha,\beta}$.

- It preserves the degrees of the polynomials of degree $\leq n$.

The first properties are consequences of the definition. The last one follows after the following proposition since $D_q x^p = [p] x^{p-1}$.

Proposition 1 *If f verifies the condition $C(\alpha)$, we have :*

$$D_q M_{n,q}^{\alpha,\beta} f(x) = \frac{[n]_q}{[n + \alpha + \beta + 2]_q} q^{\alpha+\beta+2} M_{n-1,q}^{\alpha+1,\beta+1} \left(D_q f \left(\frac{\cdot}{q} \right) \right) (qx), x \in [0, 1], \quad (5)$$

where the q -derivative of a function f is $D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$ if $x \neq 0$.

(When f' is continuous on $[0, 1]$, the limit of formula (5) is, when q tends to 1,

$$\left(M_{n,1}^{\alpha,\beta} f \right)'(x) = n(n + \alpha + \beta + 2)^{-1} M_{n-1,1}^{\alpha+1,\beta+1} (f')(x) \text{ (cf. [4])}.$$

Proof. We compute $Db_{n,k}(x) = [n] (b_{n-1,k-1}(qx)/q^{k-1} - b_{n-1,k}(qx)/q^k)$ if $1 \leq k \leq n-1$ and $Db_{n,0}(x) = -[n] b_{n-1,0}(qx)$, $Db_{n,n}(x) = [n] b_{n-1,n-1}(qx)/q^{n-1}$ to get

$$DM_{n,q}^{\alpha,\beta} f(x) = [n] \sum_{k=0}^{n-1} b_{n-1,k}(qx) (f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta}) / q^k.$$

We denote $\psi_{n,k}^{\alpha,\beta}(t) = t^{k+\alpha} (1 - qt)_q^{n-k+\beta}$, $k = 0, \dots, n$.

Recall that the q -derivative of $g_1 g_2$ is $D_q(g_1 g_2)(x) = D_q g_1(x) g_2(qx) + g_1(x) D_q g_2(x)$.

The q -Beta functions are $B_q(u, v) = \int_0^1 t^{u-1} (1 - qt)_q^{v-1} d_q t = \Gamma_q(u) \Gamma_q(v) / \Gamma_q(u + v)$.

The function $\psi_{n-1,k}^{\alpha+1,\beta+1}(\frac{t}{q}) f(q^{\beta+1} t)$, $t \in]0, 1[$, extended by 0 in 0 is continuous at 0.

Hence we may use a q -integration by parts to write, for $k = 0, \dots, n-1$:

$$\begin{aligned} B_q(k + \alpha + 2, n - k + \beta + 1) [n + \alpha + \beta + 2] (f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta}) = \\ -q^{k+\alpha} \int_0^1 (D\psi_{n-1,k}^{\alpha+1,\beta+1})\left(\frac{t}{q}\right) f(q^{\beta+1}t) d_q t = \int_0^1 q^{k+\alpha+\beta+2} \psi_{n-1,k}^{\alpha+1,\beta+1}(t) (Df)(q^{\beta+1}t) d_q t \\ - \left[\psi_{n-1,k}^{\alpha+1,\beta+1}\left(\frac{t}{q}\right) f(q^{\beta+1}t) \right]_0^1. \end{aligned}$$

Hence $(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta}) = \frac{q^{\alpha+\beta+2+k} \int_0^1 t^{k+\alpha+1} (1-qt)^{n-k+\beta} (Df)(q^{\beta+1}t) d_q t}{[n+\alpha+\beta+2] B_q(k+\alpha+2, n-k+\beta+1)}$ and

$$DM_n^{\alpha,\beta} f(x) = \frac{[n] q^{\alpha+\beta+2}}{[n+\alpha+\beta+2]} \sum_{k=0}^{n-1} \frac{\langle b_{n-1,k}, Df(\frac{\cdot}{q}) \rangle_q^{\alpha+1,\beta+1}}{B_q(k+\alpha+2, n-k+\beta+1)} b_{n-1,k}(x). \quad \blacksquare$$

Theorem 1 *The following equality holds for any $x \in [0, 1]$:*

$$M_{n,q}^{\alpha,\beta} f(x) = \sum_{j=0}^{\infty} \Phi_{j,n,q}^{\alpha,\beta}(x) f(q^{j+\beta+1}) \quad (6)$$

$$\begin{aligned} \text{where, } \Phi_{j,n,q}^{\alpha,\beta}(x) &= u_j \sum_{k=0}^n v_k b_{n,k,q}(q^{j+\beta+1}) b_{n,k,q}(x), \\ u_j &= (1-q) q^{j(\alpha+1)} (1-q^{j+1})_q^\beta, \quad j \in \mathbb{N}, \\ v_k^{-1} &= q^{k(\beta+1)} \begin{bmatrix} n \\ k \end{bmatrix}_q B_q(k+\alpha+1, n-k+\beta+1), \quad k = 0, \dots, n. \end{aligned}$$

Moreover, for any $r \in \mathbb{N}$, the sequence $\Phi_{r,n,q}^{\alpha,\beta}, \Phi_{r-1,n,q}^{\alpha,\beta}, \dots, \Phi_{0,n,q}^{\alpha,\beta}$ is totally positive, that is to say, the collocation matrix $\left(\Phi_{r-j,n,q}^{\alpha,\beta}(x_i) \right)_{i=1,\dots,m, j=0,\dots,r}$ is totally positive for any family (x_i) , $0 \leq x_1 < \dots < x_m \leq 1$.

Proof. We set $\Phi_j = \Phi_{j,q}^{\alpha,\beta}$, $b_{n,k,q} = b_k$, $c = q^{\beta+1}$. The formulae (6) come by writing the definite q -integrals $\langle b_k, f \rangle_q^{\alpha,\beta}$ as discrete sums in (4) and the Beta integrals $\langle b_k, 1 \rangle_q^{\alpha,\beta} = \begin{bmatrix} n \\ k \end{bmatrix}_q B_q(k+\alpha+1, n-k+\beta+1)$, $k = 0, \dots, n$.

For the total positivity of the Φ_j , we have to prove that, for any $m \in \mathbb{N}$ and any two families $(x_i)_{i=1,\dots,m}$, $(j_k)_{k=1,\dots,m}$, with $0 \leq x_1 \leq \dots \leq x_m \leq 1$, $0 \leq j_m \leq \dots \leq j_1$, the determinant $\det(\Phi_{j_l}(x_i))_{i=1,\dots,m, l=1,\dots,m}$ is non negative. From the multilinearity of

the determinants, there is a basic composition formula for the discrete sums (cf. [8]).

We have $\det(\Phi_{j_l}(x_i))_{i=1,\dots,m,l=1,\dots,m} = \prod_{l=1}^m u_{j_l} E$ where

$$\begin{aligned} E &= \det(\sum_{k=0}^n v_k b_k(cq^{j_l}) b_k(x_i))_{i=1,\dots,m,l=1,\dots,m} \\ &= \sum_{k_1=0}^n \cdots \sum_{k_m=0}^n v_{k_1} \cdots v_{k_m} \det(b_{k_i}(cq^{j_l}) b_{k_i}(x_i))_{i=1,\dots,m,l=1,\dots,m} \\ &= \sum_{k_1=0}^n \cdots \sum_{k_m=0}^n v_{k_1} \cdots v_{k_m} b_{k_1}(x_1) \cdots b_{k_m}(x_m) \det(b_{k_i}(cq^{j_l}))_{i=1,\dots,m,l=1,\dots,m} \\ &= \sum_{0 \leq k_1 \leq \dots \leq k_m \leq n} v_{k_1} \cdots v_{k_m} \det(b_{k_l}(x_i))_{i=1,\dots,m,l=1,\dots,m} \det(b_{k_i}(cq^{j_l}))_{i=1,\dots,m,l=1,\dots,m}. \end{aligned}$$

We know that the q -Bernstein basis is totally positive (cf. [6]). Hence we have

$\det(b_{k_l}(x_i))_{i=1,\dots,m,l=1,\dots,m} \geq 0$ and also $\det(b_{k_i}(cq^{j_l}))_{i=1,\dots,m,l=1,\dots,m} \geq 0$, since $cq^{j_1} < cq^{j_2} < \dots < cq^{j_m}$. So E is non negative and the result follows. ■

Corollary 1 *The number of sign changes of the polynomial $M_{n,q}^{\alpha,\beta} f$ on $]0, 1[$ is not greater than the number of sign changes of the function f .*

Proof. For any $r \in \mathbb{N}$, the sequence $\Phi_r, \Phi_{r-1}, \dots, \Phi_0$ is totally positive. We deduce that the number of sign changes of the polynomial $\sum_{j=0}^r \Phi_j(x) f(q^{j+\beta+1})$ is not greater than the number of sign changes of the sequence $f(q^{j+\beta+1})$, $j = 0, \dots, r$, hence not greater than the number of sign changes of the function f in $]0, 1[$ (cf. [5]).

When r tends to infinity this property is preserved hence is true for $M_n f$. ■

Corollary 2 *Let f be a function satisfying the condition $C(\alpha)$.*

1. *If f is increasing (respectively decreasing), then the function $M_{n,q}^{\alpha,\beta} f$ is increasing (respectively decreasing).*
2. *If f is convex, then the function $M_{n,q}^{\alpha,\beta} f$ is convex.*

Proof. 1) If f is monotone, for any $s \in \mathbb{R}$ the function $f - s$ has at most one sign change. Hence $M_n^{\alpha,\beta}(f - s) = M_n^{\alpha,\beta}f - s$ has at most one sign change and $M_n^{\alpha,\beta}f$ is monotone. If f is increasing, $Df(\frac{\cdot}{q})$ is positive on $]0, q[$. Since the operators $M_n^{\alpha,\beta}$ are positive, we obtain for $x \in [0, 1]$, $M_{n-1}^{\alpha+1,\beta+1} \left(Df \left(\frac{\cdot}{q} \right) \right) (qx) \geq 0$, and, using (5), $DM_n^{\alpha,\beta}f(x) \geq 0$. So the function $M_n^{\alpha,\beta}f$ is increasing.

2) Let suppose the function f is convex. Since $M_n^{\alpha,\beta}$ preserves the degree of the polynomials, for any real numbers γ_1, γ_2 there exist δ_1, δ_2 and a function g such that $g(x) = f(x) - \delta_1x - \delta_2$ and $M_n^{\alpha,\beta}f(x) - \gamma_1x - \gamma_2 = M_n^{\alpha,\beta}g(x)$. The number of sign changes of g being at most two, it is the same for $M_n^{\alpha,\beta}g$. Hence $M_n^{\alpha,\beta}f$ is convex or concave. Moreover, if a function φ is convex (respectively concave), $D^2\varphi(x) = \frac{q^3}{(q-1)^2x^2}(\varphi(q^2x) - [2]\varphi(qx) + q\varphi(x))$ is ≥ 0 (respectively ≤ 0). Hence $M_{n-2}^{\alpha+2,\beta+2} \left(D^2\varphi \left(\frac{\cdot}{q} \right) \right) (q^2x) \geq 0$. Using (5) two times we obtain $D^2M_n^{\alpha,\beta}\varphi(x) \geq 0$ and $M_n^{\alpha,\beta}\varphi$ is not concave. ■

3 Convergence properties

Theorem 2 *If f is continuous on $[0, 1]$,*

$$\|M_{n,q}^{\alpha,\beta}f - f\|_\infty \leq C_{\alpha,\beta} \omega \left(f, \frac{1}{\sqrt{[n]_q}} \right),$$

where $\|f\|_\infty$ is the uniform norm of f on $[0, 1]$ and $\omega(f, \cdot)$ is the usual modulus of continuity of f , the constant $C_{\alpha,\beta}$ being independent of n, q, f .

Proof. As $M_n^{\alpha,\beta}$ is positive, O. Shisha and B. Mond theorem can be applied. It is sufficient to prove that the order of approximation of f by $M_n^{\alpha,\beta}f$ is $O(\frac{1}{[n]})$ for the

functions $f_i(x) = x^i$, $i = 0, 1, 2$. We compute the polynomials $M_n^{\alpha, \beta} f_i$, $i = 1$ and 2 , with the help of (5) by q -integrations.

$$[n + \alpha + \beta + 2] (M_n^{\alpha, \beta} f_1(x) - x) = q^{\beta+1} [\alpha + 1] - x [\alpha + \beta + 2] \text{ and}$$

$$[n + \alpha + \beta + 2] [n + \alpha + \beta + 3] (M_n^{\alpha, \beta} f_2(x) - x^2) =$$

$$[n] [2] x (q^{\alpha+2\beta+3} [\alpha+2] (1-x) - [\beta+1] x) + [a+\beta+3] [\alpha+\beta+2] x^2 + q^{2\beta+2} [\alpha+2] [\alpha+1].$$

The result follows since $0 < q < 1$ and $0 \leq [a] \leq \max(a, 1)$ if $a \geq 0$. ■

Remark 1

In order to have uniform convergence for all continuous functions on $[0, 1]$, it is sufficient to have $\lim_{n \rightarrow \infty} M_{n,q}^{\alpha, \beta} f_i = f_i$ for $i = 1, 2$, hence $\lim_{n \rightarrow \infty} 1/[n]_q = 0$. This is realized if and only if $q = q_n$ and $\lim_{n \rightarrow \infty} q_n = 1$. Indeed, for every $n \in \mathbb{N}$, in both cases $q^n < 1/2$ and $q^n \geq 1/2$, we have $1 - q < 1/[n]_q \leq 2 \max(1 - q, \ln 2/n)$. To maximize the order of approximation by the operator $M_{n,q_n}^{\alpha, \beta}$, we are interested to have $[n]_{q_n}$ of the same order as n , that is to say to have $\rho n < [n]_{q_n} \leq n$, for some $\rho > 0$, property which holds with the following property S for (q_n) .

Definition 2 *The sequence $(q_n)_{n \in \mathbb{N}}$, has the property S if and only if there exists $N \in \mathbb{N}$ and $c > 0$ such that, for any $n > N$, $1 - q_n < c/n$.*

Lemma 1

The property S holds if and only if the property P_1 (respectively P_2) holds where :

P_1 is "There exists $N_1 \in \mathbb{N}$ and $c_1 > 0$ such that, for any $n > N_1$, $[n]_{q_n} \geq c_1 n$ ",

P_2 is "There exists $N_2 \in \mathbb{N}$ and $c_2 > 0$ such that, for any $n > N_2$, $q_n^n \geq c_2$ ".

Proof. For any $n \in \mathbb{N}$, the function $\xi(x) = (1 - x^n)/(1 - x)$ is increasing on $[0, 1[$. If S holds, we have, for any $n > N_1 = N$, $[n]_{q_n} = \xi(q_n) \geq \xi(1 - c/n) \geq n(1 - e^{-c})/c$

and P_1 follows. If P_1 holds, we have, for any $n > N = N_1$, $1/(1-q_n) \geq [n]_{q_n} \geq c_1 n$ and S follows. If P_2 holds, we have, for any $n > N = N_2$, $n(1-q_n) \leq -n \ln q_n < -\ln c_2$ and S follows. If S holds, there exists $N_2 > N$ such that, if $n > N_2$, $1-q_n < 1/2$ hence $q_n^n > e^{-2n(1-q_n)} > e^{-2c}$ and P_2 follows. ■

Theorem 3 *If the function f is continuous at the point $x \in]0, 1[$, then,*

$$\lim_{n \rightarrow \infty} M_{n, q_n}^{\alpha, \beta} f(x) = f(x) \quad (7)$$

in the two following cases :

1. *If f is bounded on $[0, 1]$ and the sequence (q_n) is such that $\lim_{n \rightarrow \infty} q_n = 1$,*
2. *If there exist real numbers $\alpha', \beta' \geq 0$ and a real $\kappa' > 0$ such that, for any $x \in]0, 1[$, $|x^{\alpha'}(1-x)^{\beta'} f(x)| \leq \kappa'$, $\alpha' < \alpha + 1$, $\beta' < \beta + 1$ and the sequence (q_n) owns the property S .*

Theorem 4 *If the function f admits a second derivative at the point $x \in]0, 1[$ then,*

in the cases 1 and 2 of theorem 3, we have the Voronovskaya-type limit :

$$\lim_{n \rightarrow \infty} [n]_{q_n} (M_{n, q_n}^{\alpha, \beta} f(x) - f(x)) = \frac{d}{dx} (x^{\alpha+1}(1-x)^{\beta+1} f'(x)) / x^{\alpha}(1-x)^{\beta}. \quad (8)$$

(The limit operator is the Jacobi differential operator for the weight $x^{\alpha}(1-x)^{\beta}$)

For the proofs of theorems 3 and 4 we need some preparation.

Proposition 2 *We set, for any $n, m \in \mathbb{N} - \{0\}$ and $x \in [0, 1]$, $q \in [1/2, 1[$,*

$$T_{n, m, q}(x) = \sum_{k=0}^n b_{n, k, q}(x) \frac{\int_0^1 t^{k+\alpha} (1-qt)_q^{n-k+\beta} (x-t)^m d_q t}{\int_0^1 t^{k+\alpha} (1-qt)_q^{n-k+\beta} d_q t}. \quad (9)$$

For any m , there exists a constant $K_m > 0$, independent of n and q , such that :

$$\sup_{x \in [0,1]} |T_{n,m,q}(x)| \leq \begin{cases} K_m / [n]_q^{m/2} & \text{if } m \text{ is even,} \\ K_m / [n]_q^{(m+1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

To prove this proposition we consider the lemmas 2 and 3.

Lemma 2 We set, for any $n, m \in \mathbb{N}$ and $x \in [0, 1]$,

$$T_{n,m,q}^1(x) = \sum_{k=0}^n b_{n,k,q}(x) \frac{\int_0^1 t^{k+\alpha} (1-qt)_q^{n-k+\beta} (x-t)_q^m d_q t}{\int_0^1 t^{k+\alpha} (1-qt)_q^{n-k+\beta} d_q t}.$$

The following recursion formula holds for any $q \in [1/2, 1[$ and $m \geq 2$:

$$\begin{aligned} [n+m+\alpha+\beta+2]_q q^{-\alpha-2m-1} T_{n,m+1,q}^1(x) = \\ (-x(1-x)D_q T_{n,m,q}^1(x) + T_{n,m,q}^1(x)(p_{1,m}(x) + x(1-q)[n+\alpha+\beta]_q [m+1]_q q^{1-\alpha-m}) \\ + T_{n,m-1,q}^1(x)p_{2,m}(x) + T_{n,m-2,q}^1(x)p_{3,m}(x)(1-q), \end{aligned} \quad (10)$$

where the polynomials $p_{i,m}(x)$, $i = 1, 2$ and 3 are uniformly bounded with regard to n and q .

Proof. 1) We set $\psi_k(t) = t^{k+\alpha}(1-qt)_q^{n-k+\beta}$ and $l_k(x) = b_{n,k}(x) / \int_0^1 \psi_k(t) d_q t$,

$k = 0, \dots, n$ and $T_{n,m,q}^1 = T_m^1$.

We compute $x(1-x)D_q T_m^1(x) = x(1-x)[m] \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(x-t)_q^{m-1} d_q t$
 $+ \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(qx-t)_q^m ([k] - [n]x) d_q t = A + B.$

We have $A = x(1-x)[m] T_{m-1}^1(x)$ and

$$\begin{aligned} B &= q^{-\alpha} \sum_{k=0}^n l_k(x) \int_0^1 (D_q \psi_k)(t) t(1-qt)(qx-t)_q^m d_q t \\ &- q^{1-\alpha-m} [n+\alpha+\beta] \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(qx-t)_q^{m+1} d_q t \\ &+ (x([n+\alpha+\beta] q^{2-\alpha-m} - [n]) + [-\alpha]) \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(qx-t)_q^m d_q t \\ &= B_1 + B_2 + B_3. \end{aligned}$$

We q -integrate by parts, setting $\sigma(t) = \left(\frac{t}{q}(1-t)(qx - \frac{t}{q})_q^m\right)$. The q -integral in B_1 becomes $\int_0^1 D_q \psi_k(t) t(1-qt)(qx - t)_q^m d_q t = [\psi_k(t)\sigma(t)]_0^1 - \int_0^1 \psi_k(t)(D_q \sigma)(t) d_q t$ for each $k = 0, \dots, n$,

$$\begin{aligned} & \text{We expand } \sigma(t) = q^{-2m}(x - \frac{t}{q})_q^{m+2} + q^{-2m}([3] - q^{m+2})x - q^m(x - \frac{t}{q})_q^{m+1} \\ & + q^{-2m+1}(x(q^{m-1} + [m](1-q)q^m) - x^2(1 + [2]q(1-q)[m]))(x - \frac{t}{q})_q^m \\ & - q^{2m+3}x^2[m](1-q)(q^{m-2} - x)(x - \frac{t}{q})_q^{m-1}. \end{aligned}$$

$$\begin{aligned} & \text{We obtain } B_1 = -q^{-\alpha-2m-1}([m+2]T_{m+1}^1(x) - [m+1]([3]x - q^{m+2}x - q^m)T_m^1(x) \\ & - q^{-\alpha-2m}[m]x(q^{m-1} + (1-q)[m]q^m - x(1+q(1-q)[2][m]))T_{m-1}^1 \\ & + q^{-\alpha+2-2m}[m-1]x^2[m](q^{m-2} - x)(1-q)T_{m-2}^1(x). \end{aligned}$$

$$\begin{aligned} & \text{Moreover we have } B_2 = -q^{1-\alpha-m}[n+\alpha+\beta](T_{n,m+1}^1(x) - (1-q)[m+1]xT_{n,m}^1(x)), \\ & B_3 = (x(q^n[\beta-m+2] - [2-\alpha-m]) + [-\alpha])(T_{n,m}^1(x) - (1-q)[m]xT_{n,m-1}^1(x)). \blacksquare \end{aligned}$$

Lemma 3 For any $m \in \mathbb{N}$, $q \in [1/2, 1[$, $x \in [0, 1]$, the expansion of $(x-t)^m$ on the Newton basis at the points x/q^i , $i = 0, \dots, m-1$ is:

$$(x-t)^m = \sum_{k=1}^m d_{m,k}(1-q)^{m-k}(x-t)_q^k, \quad (11)$$

where the coefficients $d_{m,k}$, verify $|d_{m,k}| \leq d_m$, $k = 1, \dots, m$, and d_m does not depend on x, t, q .

Proof. For $m = 1$, it is obvious. If for some $m \geq 1$, the relation (11) holds, we write $x-t = q^{-k}((x-q^k t) - (1-q)[k]x)$ for $k = 1, \dots, m$ and we obtain $(x-t)^{m+1} = \sum_{k=1}^{m+1} d_{m+1,k}(1-q)^{m+1-k}(x-t)_q^k$ with $d_{m+1,k} = q^{-k}(qd_{m,k-1} - [k]d_{m,k})$ if $k = 1, \dots, m$ and $d_{m+1,m+1} = q^{-m}d_{m,m}$. Since $|d_{m,k}| \leq d_m$ we have $|d_{m+1,k}| \leq d_{m+1} = 2^{-m}(m+1)d_m$, $k = 1, \dots, m+1$. \blacksquare

Proof of the proposition 2

At first we prove that, for any x , $|T_m^1(x)| \leq H_m/[n]^{m/2}$ if m is even (respectively $\leq H_m/[n]^{(m+1)/2}$ if m is odd), where H_m does not depend on n, q, x . We have $T_{n,0}^1(x) = 1$ and the formulae for $M_n^{\alpha,\beta} f_i, i = 1, 2$ of the proof of theorem 2 give the result for $m = 1$ and 2. The product $[n + \alpha + \beta](1 - q) = 1 - q^{n+\alpha+\beta}$ is positive and bounded by $\max(1, |1 - 2^{-(1+\alpha+\beta)}|)$. If the result is true for some $p \geq 2$, p odd (respectively even) and any $m \leq p$, the result for $p + 1$ follows from the recursion formula (10) of lemma 2.

Then, we write, for any $n, m \in \mathbb{N}$ and $x \in [0, 1]$, using lemma 3, and $1 - q < 1/[n]$,

$$|T_{n,m,q}(x)| \leq d_m \sum_{k=1}^m (1 - q)^{m-k} |T_k^1(x)| \leq d_m (T_m^1(x) + \sum_{k=1}^{m-1} [n]^{-m+k} H_k [n]^{-k/2})$$

$$\leq d_m T_m^1(x) + \sum_{k=1}^{m-1} H_k [n]^{-(m+1)/2}) \text{ and the result follows.} \blacksquare$$

Now the following lemma is the key.

Lemma 4 *Let (q_n) be a sequence owning the property S, $x \in]0, 1[$ and $\delta \in]0, 1[$, $\delta < \min(x, 1 - x)$. Let $\alpha, \beta, \alpha', \beta'$ be real numbers such that $\alpha', \beta' \geq 0$, $\alpha > \alpha' - 1$, $\beta > \beta' - 1$. We set $\varphi(t) = t^{-\alpha'}(1 - t)^{-\beta'}$, $t \in]0, 1[$ and $I_{x,\delta}(t) = 1$ if $|t - x| > \delta$, $I_{x,\delta}(t) = 0$ elsewhere.*

The sequence $E_n(x, \delta) = \sum_{k=0}^n b_{n,k,q_n}(x) \frac{\int_0^1 t^{k+\alpha} (1 - q_n t)_{q_n}^{n-k+\beta} \varphi(q_n^{\beta+1} t) I_{x,\delta}(t) d_{q_n} t}{\int_0^1 t^{k+\alpha} (1 - q_n t)_{q_n}^{n-k+\beta} d_{q_n} t}$ verifies $\lim_{n \rightarrow \infty} n E_n(x, \delta) = 0$ for any x and δ such that $0 < \delta < x < 1 - \delta$.

Proof. Let $\bar{\alpha}$ (respectively $\bar{\beta}$) be the smallest integer such that $\bar{\alpha} \geq \alpha$ (respectively $\bar{\beta} \geq \beta$) and τ (respectively τ') be a real number such that $\tau > \frac{\bar{\alpha} + \bar{\beta} + 2}{\alpha - \alpha' + 1}$ (respectively $\tau' > \frac{\bar{\alpha} + \bar{\beta} + 2}{\beta - \beta' + 1}$).

For any $k = 0, \dots, n$, we have $\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta} d_q t \geq \int_0^1 t^{k+\bar{\alpha}}(1-qt)_q^{n-k+\bar{\beta}} d_q t = \frac{[k+\bar{\alpha}]![n-k+\bar{\beta}]!}{[n+\bar{\alpha}+\bar{\beta}+1]!} \geq [k]^{-1}(n+\bar{\alpha}+\bar{\beta}+1)^{-(\bar{\alpha}+\bar{\beta}+1)}$. We set $I_{x,\delta}^-(t) = 1$ if $0 < t < x - \delta$ and $I_{x,\delta}^+(t) = 1$ if $x + \delta < t < 1$, $I_{x,\delta}^-(t) = I_{x,\delta}^+(t) = 0$ elsewhere.

We split the interval $(0, 1)$ introducing $e_n = 1/n^\tau$, $e'_n = 1 - 1/n^{\tau'}$, $n \in \mathbb{N}$, and we define, using again $\psi_{n,k,q}^{\alpha,\beta}(t) = t^{k+\alpha}(1-qt)_q^{n-k+\beta}$ and $l_{n,k,q}(x) = b_{n,k,q}(x) / \int_0^1 \psi_{n,k,q}^{\alpha,\beta}(t) d_q t$ of lemma 2, $A_n^1 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^{e_n} t^{k+\alpha-\alpha'}(1-q_n t)_{q_n}^{n-k+\beta} d_{q_n} t$,

$$A_n^2 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_{e_n}^1 t^{k+\alpha-\alpha'}(1-q_n t)_{q_n}^{n-k+\beta} I_{x,\delta}^-(t) d_{q_n} t,$$

$$A_n^3 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^{e'_n} t^{k+\alpha}(1-q_n t)_{q_n}^{n-k+\beta} / (1-q_n^{\beta+1} t)^{\beta'} I_{x,\delta}^+(t) d_{q_n} t,$$

$$A_n^4 = \sum_{k=0}^n l_{n,k,q_n}(x) \int_{e'_n}^1 t^{k+\alpha}(1-q_n t)_{q_n}^{n-k+\beta} / (1-q_n^{\beta+1} t)^{\beta'} I_{x,\delta}^+(t) d_{q_n} t.$$

If $t > x$, (respectively if $t < x$) then $t^{-\alpha'} < x^{-\alpha'}$ (respectively $(1 - q_n^{\beta+1} t)^{\beta'} > (1 - q_n^{\beta+1} x)^{\beta'} \geq (1 - x)^{\beta'}$).

Hence, we have $E_n(x, \delta) \leq (1/2)^{-(\beta+1)\alpha'} ((1-x)^{-\beta'} (A_n^1 + A_n^2) + x^{-\alpha'} (A_n^3 + A_n^4))$ if $q_n \geq 1/2$, and it is sufficient to prove $\lim_{n \rightarrow \infty} n A_n^i = 0$ for $i = 1, 2, 3, 4$.

If $q_n^n \geq c$ and $e_n < 1/2$, we have for $k = 0, \dots, n$, $\int_0^{e_n} t^{k+\alpha-\alpha'}(1-q_n t)_{q_n}^{n-k+\beta} d_{q_n} t = q_n^{-k(\beta+1)} [n]_{q_n}^{-1} \int_0^{e_n} b_{n,k,q}(q_n^{\beta+1} t) t^{\alpha-\alpha'} (1-q_n t)_{q_n}^\beta d_{q_n} t \leq [n]_{q_n}^{-1} \gamma_1 e_n^{\alpha-\alpha'+1}$, where γ_1 does not depend on k, n, x , since $q_n^{(\beta+1)k} \geq c^{\beta+1}$, $0 \leq b_{n,k,q_n}(q_n^{\beta+1} t) \leq 1$, and, $(1-q_n t)_{q_n}^\beta \leq 1$ if $\beta \geq 0$ and $t \in [0, 1]$, (respectively $(1-q_n t)_{q_n}^\beta \leq (1-e_n)^{-1} \leq 2$ if $\beta < 0$ and $t \in [0, e_n]$). Hence, we have $A_n^1 \leq \gamma_1 (n + \bar{\alpha} + \bar{\beta} + 1)^{\bar{\alpha}+\bar{\beta}+1} n^{-\tau(\alpha-\alpha'+1)}$. The choice of τ and the property S on (q_n) give $\lim_{n \rightarrow \infty} n A_n^1 = 0$.

We choose $m \in \mathbb{N}$ such that $m > \tau\alpha' + 1$ and we write

$$\begin{aligned} A_n^2 &\leq n^{\tau\alpha'} \delta^{-2m} \sum_{k=0}^n l_{n,k,q_n}(x) \int_{e_n}^1 t^{k+\alpha}(1-q_n t)_{q_n}^{n-k+\beta} (x-t)^{2m} d_{q_n} t \\ &\leq n^{\tau\alpha'} \delta^{-2m} T_{n,2m,q_n}(x) \leq K_{2m} \delta^{-2m} n^{\tau\alpha'-m}, \text{ hence } \lim_{n \rightarrow \infty} n A_n^2 = 0 \text{ by the choice of } m. \end{aligned}$$

Now we have, if $t < e'_n$, $(1 - q_n^{\beta+1} t)^{\beta'} > (1 - e'_n)^{\beta'} \geq n^{-\tau'\beta'}$, hence

$A_n^3 \leq n^{\tau'\beta'} \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^{e'_n} t^{k+\alpha} (1-q_nt)_{q_n}^{n-k+\beta} I_{x,\delta}^+(t) d_{q_n} t$. We choose $m' \in \mathbb{N}$ such that $m' > \tau'\beta' + 1$ to have $A_n^3 \leq n^{\tau'\beta'} \delta^{-2m'} \sum_{k=0}^n l_{n,k,q_n}(x) \int_0^1 t^{k+\alpha} (1-q_nt)_{q_n}^{n-k+\beta} (x-t)^{2m'} d_{q_n} t$
 $\leq n^{\tau'\beta'} \delta^{-2m'} T_{n,2m',q_n} \leq K_{2m'} \delta^{-2m'} n^{\tau'\beta'-m'}$, hence $\lim_{n \rightarrow \infty} nA_n^3 = 0$ by the choice of m' .

To finish, we prove that $(1 - q_n^{\beta-\beta'+1}t)_{q_n}^{\beta'} \leq (1 - q_n^{\beta+1}t)^{\beta'}$ for any $t \in [0, 1]$.

If $0 \leq \beta' < 1$, we use the q -binomial formula (cf. [1]) and the inequalities

$[-\beta'] \leq -\beta'$ and $[-\beta' + k] / [k] \geq (-\beta' + k) / k$ for any integer $k \geq 1$. In the other cases, if l is the integer such that $l \leq \beta' < l + 1$, we use the rules of product of q -binomials to write

$$(1 - q_n^{\beta-\beta'+1}t)_{q_n}^{\beta'} = (1 - q_n^{\beta-\beta'+1}t)_{q_n}^l (1 - q_n^{\beta-\beta'+1+l}t)_{q_n}^{\beta'-l} \text{ and the result follows.}$$

Then, with the same rules, we write, for any $k = 0, \dots, n$ and $t \in [0, 1]$,

$$(1 - q_nt)_{q_n}^{n-k+\beta} = (1 - q_nt)_{q_n}^{\beta-\beta'} (1 - q_n^{\beta-\beta'+1}t)_{q_n}^{\beta'} (1 - q_n^{\beta+1}t)_{q_n}^{n-k} \text{ and}$$

$$(1 - q_nt)_{q_n}^{n-k+\beta} / (1 - q_n^{\beta+1}t)^{\beta'} \leq (1 - q_nt)_{q_n}^{\beta-\beta'} (1 - q_n^{\beta+1}t)_{q_n}^{n-k}. \text{ We deduce if } q_n^n \geq c \text{ and}$$

$$e'_n > 1/2, A_n^4 \leq \sum_{k=0}^n q_n^{-(\beta+1)k} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n}^{-1} l_{n,k,q_n}(x) \int_{e'_n}^1 t^\alpha (1 - q_nt)_{q_n}^{\beta-\beta'} b_{n,k,q_n}(q_n^{\beta+1}t) d_{q_n} t \\ \leq \gamma_2(n + \bar{\alpha} + \bar{\beta} + 1)^{\bar{\alpha}+\bar{\beta}+1} (1 - e'_n)^{\beta-\beta'+1} \text{ where } \gamma_2 \text{ does not depend on } k, n, x. \text{ The choice of } e'_n \text{ and } \tau' \text{ gives } \lim_{n \rightarrow \infty} nA_n^4 = 0. \blacksquare$$

Proof of theorem 3

Suppose f is continuous at $x \in]0, 1[$. Let $\varepsilon > 0$ be an arbitrary real number. There exists $\delta' > 0$ such that $|f(x) - f(t)| < \varepsilon$ for any $t \in [0, 1]$ such that $|x - t| < \delta'$. Let $\delta = \delta'/2$ and $N' \in \mathbb{N}$ such that $(1 - q_n^{\beta+1})x < \delta$ for $n > N'$. Then we have, if $|x - t| < \delta$ and $n > N'$, the inequalities $-\delta < -q_n^{\beta+1}\delta < x - q_n^{\beta+1}t = q_n^{\beta+1}(x - t) + (1 - q_n^{\beta+1})x < 2\delta$ and $|f(x) - f(q_n^{\beta+1}t)| < \varepsilon$.

Hence, we have, if $|f|$ is bounded by κ , $|f(x) - f(q_n^{\beta+1}t)| < \varepsilon + 2\kappa I_{x,\delta}(t)$ and, in

the case 2, $|f(x) - f(q_n^{\beta+1}t)| < \varepsilon + (|f(x)| + \kappa'(q_n^{\beta+1}t)^{-\alpha'}(1 - q_n^{\beta+1}t)^{-\beta'}I_{x,\delta}(t))$.

We apply the operator $M_{n,q_n}^{\alpha,\beta}$ at the function $h_x(t) = f(t) - f(x)$.

$$\begin{aligned} \text{We have } |M_{n,q_n}^{\alpha,\beta}f(x) - f(x)| &= |M_{n,q_n}^{\alpha,\beta}h_x(x)| \leq (M_{n,q_n}^{\alpha,\beta}|h_x|)(x) \\ &\leq \begin{cases} \varepsilon + 2\kappa T_{n,2,q_n}(x)/\delta^2 \text{ in the case 1,} \\ \varepsilon + |f(x)| T_{n,2,q_n}(x)/\delta^2 + \kappa' E_n(x, \delta, q_n) \text{ in the case 2.} \end{cases} \end{aligned}$$

The second term (respectively and the third term in the case 2) of the right hand side vanishes when $[n]_{q_n}$ tends to infinity. Since $\lim_{n \rightarrow \infty} 1/[n]_{q_n} = 0$ in both cases (remark 1), the result follows. ■

Proof of theorem 4

We write Taylor formula at the point x ,

$$f(t) = f(x) + (t - x)f'(x) + (t - x)^2 f''(x)/2 + (t - x)^2 \varepsilon(t - x) \text{ where } \lim_{u \rightarrow 0} \varepsilon(u) = 0.$$

We apply the operator $M_{n,q_n}^{\alpha,\beta}$ at the function f of the variable t to obtain

$$M_{n,q_n}^{\alpha,\beta}f(x) - f(x) = -f'(x)T_{n,1,q_n}(x) + \frac{f''(x)}{2}T_{n,2,q_n}(x) + R_n(x) \text{ where}$$

$$R_n(x) = M_{n,q_n}^{\alpha,\beta}\zeta_x(x) \text{ with } \zeta_x(t) = (t - x)^2 \varepsilon(t - x). \text{ We use } \lim_{q \rightarrow 1} [a]_q = a \text{ for any}$$

$a \in \mathbb{R}$ and we verify, with the help of the formulae of the proof of theorem 2, that

$$\lim_{[n]_{q_n} \rightarrow \infty} [n]_{q_n} T_{n,1,q_n}(x) = (\alpha + \beta + 2)x - \alpha - 1 \text{ and } \lim_{[n]_{q_n} \rightarrow \infty} [n]_{q_n} T_{n,2,q_n}(x) = 2x(1 - x).$$

So, to obtain the result, we have to prove that $\lim_{[n]_{q_n} \rightarrow \infty} [n]_{q_n} R_n(x) = 0$. We proceed in

the same manner as in the proof of theorem 3. For any arbitrary $\eta > 0$, we can find

$\delta > 0$ such that, for n great enough, $\varepsilon(x - t) < \eta$ if $|x - q_n^{\beta+1}t| < \delta$.

We obtain the inequality $|\zeta_x(t)| \leq \eta(x - t)^2 + (\rho_x + |f(t)|)I_{x,\delta}(q^{-(\beta+1)}t)$ for any $t \in]0, 1[$, where ρ_x is independent of t and δ . We deduce

$$[n]_{q_n} |R_n(x)| \leq \begin{cases} [n]_{q_n} (\eta T_{n,2,q_n}(x) + (\rho_x + \kappa)T_{n,4,q_n}(x)/\delta^4) \text{ in the case 1,} \\ [n]_{q_n} (\eta T_{n,2,q_n}(x) + \rho_x T_{n,4,q_n}(x)/\delta^4) + \kappa' n E_n(x, \delta) \text{ in the case 2.} \end{cases}$$

The right hand side tends to $2\eta x(1-x)$ when n (hence $[n]_{q_n}$) tends to infinity and is as small as wanted. ■

Remark 2

1) We see that the best order of approximation in (8) is in $1/[n]_{q_n}$. If $1-q_n = 1/n^\gamma$ with $0 < \gamma < 1$, then $\lim_{n \rightarrow \infty} [n]_{q_n}/n^\gamma = 1$, hence $[n]_{q_n}$ can be replaced by n^γ in (8). If $1-q_n = 1/n \log n$ or $1/n^\gamma$ with $\gamma > 1$, then $\lim_{n \rightarrow \infty} [n]_{q_n}/n = 1$, $[n]_{q_n}$ can be replaced by n and we refound exactly the Voronovskaya-limit property of $M_{n,1}^{\alpha,\beta}(x)$ (case 1).

2) In the case 2, the theorems 3 and 4 are valid for $M_{n,1}^{\alpha,\beta}$, if wf is Lebesgue integrable on $[0, 1]$, and this result is new. (In the proof the Jackson integrals have to be replaced by Lebesgue integrals)

Theorem 5 *If f' is continuous on $[0, 1]$ and $q > 1/2$, then*

$$\|D_q(M_{n,q}^{\alpha,\beta}f) - f'\|_\infty \leq C'_{\alpha,\beta} \left(\omega \left(f', \frac{1}{\sqrt{[n]_q}} \right) + \omega(f', 1-q) \right) + \frac{[\alpha + \beta + 2]_q}{[n]_q} \|f'\|_\infty,$$

where $C'_{\alpha,\beta}$ is a constant independent of n, q, f .

Hence, if $\lim_{n \rightarrow \infty} q_n = 1$, then $\lim_{n \rightarrow \infty} D_{q_n}(M_{n,q_n}^{\alpha,\beta}f) = f'(x)$ uniformly on $[0, 1]$.

Proof. We write, using (5), for any $x \in [0, 1]$,

$$\begin{aligned} DM_n^{\alpha,\beta}f(x) - f'(x) &= \frac{[n]}{[n+\alpha+\beta+2]} \left(M_{n-1}^{\alpha+1,\beta+1} \left(Df \left(\frac{\cdot}{q} \right) \right) (qx) - Df(x) + Df(x) - f'(x) \right) \\ &+ \left(\frac{[n]}{[n+\alpha+\beta+2]} - 1 \right) f'(x). \text{ Since } 0 < \frac{[n]}{[n+\alpha+\beta+2]} < 1, \text{ we have } |D(M_n^{\alpha,\beta}f(x)) - f'(x)| \\ &\leq \left| M_{n-1}^{\alpha+1,\beta+1} \left(Df \left(\frac{\cdot}{q} \right) \right) (qx) - Df(x) \right| + |Df(x) - f'(x)| + \frac{[\alpha+\beta+2]}{[n]} |f'(x)|. \end{aligned}$$

The theorem (2) for the function $Df \left(\frac{\cdot}{q} \right)$ gives

$$\left| M_{n-1}^{\alpha+1,\beta+1} \left(Df \left(\frac{\cdot}{q} \right) \right) (qx) - Df \left(\frac{\cdot}{q} \right) (qx) \right| \leq C_{\alpha+1,\beta+1} \omega \left(Df \left(\frac{\cdot}{q} \right), \frac{1}{\sqrt{[n-1]}} \right). \text{ Moreover}$$

$|Df(x) - f'(x)| = |f'(y) - f'(x)|$ for some y with $qx < y < x$ hence $|y - x| < 1 - q$ and $|Df(x) - f'(x)| \leq \omega(f', 1 - q)$. The modulus of continuity of $Df\left(\frac{\cdot}{q}\right)$ is linked with the modulus of continuity of f' . Indeed, for any $y_i \in [0, 1]$ and $i = 1, 2$, there exists z_i , such that $y_i < z_i < y_i/q$ and $Df\left(\frac{\cdot}{q}\right)(y_i) = f'(z_i)$. As $|z_1 - z_2| \leq |y_1 - y_2|/q + (1 - q)/q$ we get $\omega\left(Df\left(\frac{\cdot}{q}\right), t\right) = \sup_{|y_1 - y_2| < t} \left| Df\left(\frac{\cdot}{q}\right)(y_1) - Df\left(\frac{\cdot}{q}\right)(y_2) \right| \leq \sup_{|y_1 - y_2| < t} (|f'(z_1) - f'(z_2)|) \leq 2(\omega(f', t) + \omega(f', 1 - q))$ and the result follows. ■

Corollary 3 *If f' is continuous on $[0, 1]$ and $1 - q_n = o(1/n^4)$, then*

$$\lim_{n \rightarrow \infty} (M_{n, q_n}^{\alpha, \beta} f)'(x) = f'(x) \text{ uniformly on } [0, 1].$$

Proof. For any $x \in [0, 1]$, there exists $u \in (q_n x, x)$ such that

$$D_{q_n}(M_{n, q_n}^{\alpha, \beta} f)(x) = (M_{n, q_n}^{\alpha, \beta} f)'(u) \text{ and } (x - u) < 1 - q_n.$$

Hence $|D_{q_n}(M_{n, q_n}^{\alpha, \beta} f)(x) - (M_{n, q_n}^{\alpha, \beta} f)'(x)| \leq (1 - q_n) |(M_{n, q_n}^{\alpha, \beta} f)''(v)|$ for some v and $\|D_{q_n}(M_{n, q_n}^{\alpha, \beta} f) - (M_{n, q_n}^{\alpha, \beta} f)'\|_{\infty} \leq n^4(1 - q_n) \|M_{n, q_n}^{\alpha, \beta} f\|_{\infty} \leq n^4(1 - q_n) \|f\|_{\infty}$, via Markov inequality. ■

4 self-adjointness properties

In this part $q \in]0, 1[$ is independent of n .

On the space of polynomials $\langle \cdot, \cdot \rangle_q^{\alpha, \beta}$ is an inner product. Let $(P_{r, q}^{\alpha, \beta})_{r \in \mathbb{N}}$ be the sequence of the orthogonal polynomials for $\langle \cdot, \cdot \rangle_q^{\alpha, \beta}$ such that degree of $P_{r, q}^{\alpha, \beta} = r$ and $P_{r, q}^{\alpha, \beta}(0) = \begin{bmatrix} r + \alpha \\ r \end{bmatrix}_q = \frac{[\alpha + r]_q \cdots [\alpha + 1]_q}{[r]_q!}$. We set $\nu_r = \left(\langle P_{r, q}^{\alpha, \beta}, P_{r, q}^{\alpha, \beta} \rangle_q^{\alpha, \beta} \right)^{1/2}$.

We define $U_q^{\alpha, \beta}$ which is a q -analogue of the Jacobi differential operator by:

$$U_q^{\alpha, \beta} f(x) = D_q \left(x^{\alpha+1} (1 - q^{-\beta-1} x)_q^{\beta+1} D_q f\left(\frac{x}{q}\right) \right) / x^{\alpha} (1 - q^{-\beta} x)_q^{\beta}. \quad (12)$$

Proposition 3 *The operator $U_q^{\alpha,\beta}$ is self-adjoint for $\langle \cdot, \cdot \rangle_q^{\alpha,\beta}$. It preserves the space of polynomials of degree $r \in \mathbb{N}$. Consequently, for any $r \in \mathbb{N}$, $P_{r,q}^{\alpha,\beta}$ is eigenvector of $U_q^{\alpha,\beta}$ for the eigenvalue $\mu_r^{\alpha,\beta} = -q^{-\beta-r} [r]_q [r + \alpha + \beta + 1]_q$.*

Proof.

$U_q^{\alpha,\beta}$ is a q -differential operator of order 2. We compute with q -binomial relations, $U_q^{\alpha,\beta} f(x) = (-q^{\alpha-\beta} [\beta + 1] x + [\alpha + 1] (1 - q^{-\beta-1} x)) Df(x) + (1 - q^{-\beta-1} x) \frac{x}{q} D^2 f(\frac{x}{q})$, hence the operator U preserves the degree of polynomials. If f and g are polynomials $\langle Uf, g \rangle$ is well defined. We write, since the q -integration by parts is valid, $\langle U_q^{\alpha,\beta} f, g \rangle = \left[x^{\alpha+1} (1 - q^{-\beta-1} x)_q^{\beta+1} Df(\frac{x}{q}) g(x) \right]_0^{q^{\beta+1}} - \int_0^{q^{\beta+1}} (qx)^{\alpha+1} (1 - q^{-\beta} x)_q^{\beta+1} Df(x) Dg(x) d_q x$ and the first term vanishes. We compute $U_q^{\alpha,\beta} f(x)$ for $f(x) = x^r$ to obtain $U_q^{\alpha,\beta} f(x) = q^{1-r} [r] ([\alpha + r] x^{r-1} (1 - x) - q^{-\beta-1} [\beta + 1] x^r)$ where the coefficient of x^r is the eigenvalue $\mu_r^{\alpha,\beta}$. ■

Proposition 4 *The eigenvectors of the operators $M_{n,q}^{\alpha,\beta}$, $n \in \mathbb{N}$, are the polynomials*

$P_{r,q}^{\alpha,\beta}$, $r \in \mathbb{N}$ and, if f satisfies $c(\alpha)$, we have

$$M_{n,q}^{\alpha,\beta} f = \sum_{r=0}^n \lambda_{n,r}^{\alpha,\beta} \langle f, P_{r,q}^{\alpha,\beta} \rangle_q^{\alpha,\beta} P_{r,q}^{\alpha,\beta} / \nu_r^2 \text{ with the eigenvalues}$$

$$\lambda_{n,r}^{\alpha,\beta} = q^{r(r+\alpha+\beta+1)} \frac{[n]!}{[n-r]!} \frac{\Gamma_q(n + \alpha + \beta + 2)}{\Gamma_q(n + r + \alpha + \beta + 2)} \text{ if } r \leq n, \lambda_{n,r}^{\alpha,\beta} = 0 \text{ otherwise.}$$

Proof. Since M_n is self adjoint and preserve the degree of polynomials, the orthogonal polynomials P_r are eigenvectors. The eigenvalue $\lambda_{n,r}^{\alpha,\beta}$ is obtained by computing $M_n f(x)$ for $f(x) = x^r$.

$$\text{We use (5) } r \text{ times to get } D^r M_n f(x) = \frac{q^{r(\alpha+\beta+r+1)} [r]! [n] \dots [n-r+1]}{[n+\alpha+\beta+2] \dots [n+r+\alpha+\beta+1]} \cdot \blacksquare$$

Corollary 4 1. *For any $n, m \in \mathbb{N}$, the operators $M_{n,q}^{\alpha,\beta}$ and $M_{m,q}^{\alpha,\beta}$ commute on the space of functions satisfying $C(\alpha)$.*

2. For any $n \in \mathbb{N}$, the operators $M_{n,q}^{\alpha,\beta}$ and $U_q^{\alpha,\beta}$ commute on the space of functions f such that f' is defined in a neighborhood of 0 and is continuous at the point 0.

Proof. 2) For any $r \in \mathbb{N}$ the q -integrals $\langle f, UP_r \rangle$ and $\langle Uf, P_r \rangle$ are well defined if f' is continuous at the point 0. We go from one to the other by two q -integrations by parts which are valid because $\lim_{x \rightarrow 0} Df(\frac{x}{q}) = f'(0)$. Then we write $UM_n f = \sum_{r=0}^n \lambda_{n,r} \langle f, P_r \rangle \mu_r P_r / \nu_r^2 = \sum_{r=0}^n \lambda_{n,r} \langle Uf, P_r \rangle P_r / \nu_r^2 = M_n Uf$. ■

Remark 3

This proposition and its corollary open a field to study $\lim_{n \rightarrow \infty} M_{n,q}^{\alpha,\beta} f$ for q fixed. Formally we have $\lim_{n \rightarrow \infty} M_{n,q}^{\alpha,\beta} f = S_q^{\alpha,\beta} f = \sum_{r=0}^{\infty} q^{r(r+\alpha+\beta+1)} \langle f, P_{r,q}^{\alpha,\beta} \rangle_q^{\alpha,\beta} P_{r,q}^{\alpha,\beta} / \nu_r^2$ and $\lim_{n \rightarrow \infty} M_{n,q}^{\alpha,\beta} Q = \sum_{r=0}^{\deg Q} q^{r(r+\alpha+\beta+1)} \langle Q, P_{r,q}^{\alpha,\beta} \rangle_q^{\alpha,\beta} P_{r,q}^{\alpha,\beta}(x) / \nu_r^2$ if Q is a polynomial. So $\lim_{n \rightarrow \infty} M_{n,q}^{\alpha,\beta} f = f$, if and only if f is a constant. Moreover, $\lambda_{n-1,r}^{\alpha,\beta} - \lambda_{n,r}^{\alpha,\beta} = \frac{q^{n+\beta} \lambda_{n,r}^{\alpha,\beta} \mu_r^{\alpha,\beta}}{[n][n+\alpha+\beta+1]}$, $r \in \mathbb{N}$, hence $M_{n-1}^{\alpha,\beta} f - M_n^{\alpha,\beta} f = \frac{q^{n+\beta}}{[n][n+\alpha+\beta+1]} U_q^{\alpha,\beta} M_n^{\alpha,\beta} f$ and it is easy to prove (cf. [2]) that, when f' is defined in a neighborhood of 0, continuous at 0, $\|M_{n,q}^{\alpha,\beta} f - S_q^{\alpha,\beta} f\|_{\infty} \leq \gamma_n \sup_{x \in [0, q^{\beta+2}]} |U_q^{\alpha,\beta} f(x)|$, where $\gamma_n = \sum_{k=n+1}^{\infty} \frac{q^{k+\beta}}{[k]_q [k+\alpha+\beta+1]_q} \sim \frac{q^{n+\beta+1}}{[n]_q}$. Of course $U_q^{\alpha,\beta} f$ has to be bounded on $[0, q^{\beta+2}]$, which is true, for example, if f is bounded on $[0, 1]$ and continuous on $[0, A]$ for some $A < 1$.

Proposition 5 The polynomials $P_{r,q}^{\alpha,\beta}$ are q -extensions of Jacobi polynomials for the weight $x^{\alpha}(1-x)^{\beta}$ denoted $P_r^{\alpha,\beta}$, $r \in \mathbb{N}$. They own the following properties which are the q -analogues of the well-known properties of Jacobi polynomials.

1. For any $r \in \mathbb{N}$, $\lim_{q \rightarrow 1} P_{r,q}^{\alpha,\beta} = P_r^{\alpha,\beta}$,

2. For any $r \in \mathbb{N}$, the polynomial $P_{r,q}^{\alpha,\beta}$ is a q -hypergeometric function (cf. [7]) :

$$P_{r,q}^{\alpha,\beta}(x) = \begin{bmatrix} \alpha+r \\ r \end{bmatrix} {}_2\Phi_1 \left[\begin{matrix} q^{-r}, q^{r+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix} ; q, q^{-\beta}x \right]$$

So we have $P_{r,q}^{\alpha,\beta}(x) = \begin{bmatrix} \alpha+r \\ r \end{bmatrix} p_r(q^{-\beta-1}x; q^{\alpha+1}, q^{\beta+1} : q)$, where $p_r(x; u, v : q)$ is the shifted little q -Jacobi polynomial of degree r (cf. [1], p.592).

3. They verify a q -analogue of Rodrigues formula :

$$P_{r,q}^{\alpha,\beta}(x) = \frac{1}{[r]!} \frac{D_q^r (x^{\alpha+r} (1 - q^{-\beta-r}x)_q^{\beta+r})}{x^\alpha (1 - q^{-\beta}x)_q^\beta}.$$

4. We have the relation for the q -derivative :

$$D_q P_{r,q}^{\alpha,\beta} \left(\frac{\cdot}{q} \right) = -q^{-\beta-r} [r + \alpha + \beta + 1] P_{r-1,q}^{\alpha+1,\beta+1}.$$

Proof. 2) We look for the analytic solutions of the equation $U_q^{\alpha,\beta} f - \mu_{r,q}^{\alpha,\beta} f = 0$.

We write $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $U_q^{\alpha,\beta} f(x) - \mu_{r,q}^{\alpha,\beta} f(x) = [\alpha + 1] a_1 - \mu_{r,q}^{\alpha,\beta} a_0 + q^{-\beta} \sum_{k=1}^{\infty} ([k + 1] [k + \alpha + 1] q^\beta a_{k+1} - ([k] [k + \alpha + \beta + 1] - \mu_{r,q}^{\alpha,\beta} q^{k+\beta}) a_k) q^{-k} x^k$. We obtain $\frac{a_{k+1}}{a_k} = -q^{-r-\beta} \frac{([r] - [k])([k + r + \alpha + \beta + 1])}{[k + 1] [k + \alpha + 1]} = q^{-\beta} R(q^k)$, for any $k \in \mathbb{N}$, with $R(t) = \frac{(q^{-r} - t^{-1})(q^{r+\alpha+\beta+1} - t^{-1})}{(q - t^{-1})(q^{\alpha+1} - t^{-1})}$ and the result follows.

3) For any polynomial Q of degree $< r$, we verify, with the help of q -integrations by parts that $\langle \frac{D_q^r (x^{\alpha+1} (1 - q^{-\beta-r}x)_q^{\beta+r})}{x^\alpha (1 - q^{-\beta}x)_q^\beta}, Q \rangle_q^{\alpha,\beta} = 0$.

We compute $P_{r,q}^{\alpha,\beta}(0)$ by using a q -extension of Leibniz formula. We write $D_q^r (x^{\alpha+1} (1 - q^{-\beta-r}x)_q^{\beta+r}) = \sum_{k=0}^r \begin{bmatrix} r \\ k \end{bmatrix} \frac{\Gamma_q(\alpha+r+1) \Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+k+1) \Gamma_q(\beta+r-k+1)} x^{\alpha+k} (1 - q^{-\beta}x)^{\beta+r-k} q^{c_k}$, where $c_k = k(k + \alpha - \beta - r + (k-1)/2)$ and $D_q^r (x^{\alpha+1} (1 - q^{-\beta-r}x)_q^{\beta+r}) / x^\alpha (1 - q^{-\beta}x)_q^\beta = \sum_{k=0}^r \begin{bmatrix} r \\ k \end{bmatrix} \frac{\Gamma_q(\alpha+r+1) \Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+k+1) \Gamma_q(\beta+r-k+1)} x^k (1 - q^{-\beta-r+k}x)^{r-k} q^{c_k} = A(x)$. We obtain $A(0) = \frac{\Gamma_q(\alpha+r+1)}{\Gamma_q(\alpha+1)} = [r]! \begin{bmatrix} r+\alpha \\ r \end{bmatrix}$ hence $P_{r,q}^{\alpha,\beta}(x) = A(x) / [r]!$.

1) We take $\lim_{q \rightarrow 1} A(x) = \sum_{k=0}^r \binom{r}{k} \frac{\Gamma(\alpha+r+1)\Gamma(\beta+r+1)}{\Gamma(\alpha+k+1)\Gamma(\beta+r-k+1)} x^k (1-x)^{r-k}$. It is $r!P_r^{\alpha,\beta}(x)$ (Rodrigues formula).

4) We use (5) to prove that $D_q P_{r,q}^{\alpha,\beta}(\frac{\cdot}{q})$ is eigenvector of $M_{r-1,q}^{\alpha+1,\beta+1}$. Hence, it is equal to $P_{r-1,q}^{\alpha+1,\beta+1}$ up to a constant. We compute $D_q P_{r,q}^{\alpha,\beta}(0) = a_1 = \mu_{r,q}^{\alpha,\beta} a_0$, hence $a_1 = - \left[\begin{smallmatrix} r+\alpha \\ r \end{smallmatrix} \right] q^{-\beta-r} [r] [r + \alpha + \beta + 1] / [\alpha + 1] = -q^{-\beta-r} [r + \alpha + \beta + 1] \left[\begin{smallmatrix} r+\alpha \\ r-1 \end{smallmatrix} \right]$ and the result follows. ■

5 The case $\alpha = \beta = -1$

In this part, we study the operators $M_{n,q}^{-1,-1}$. They are built with $M_{n+1,q}^{0,0}$ as kantorovich operators are built with Bernstein operators (formula (13)).

Definition 3

The operator $M_{n,q}^{-1,-1}$ is defined by replacing $\alpha = \beta$ by -1 in formula (1). It is :

$$M_{n,q}^{-1,-1} f(x) = \sum_{k=0}^n f_{n,k,q}^{-1,-1} b_{n,k,q}(x)$$

with $f_{n,0,q}^{-1,-1} = f(0)$, $f_{n,n,q}^{-1,-1} = f(1)$ and the coefficients $f_{n,k,q}^{-1,-1}$, for $k = 1, \dots, n-1$, are given by (3) taking $\alpha = \beta = -1$.

The bilinear form is $\langle f, g \rangle_q^{-1,-1} = \int_0^1 \frac{f(t)g(t)}{t(1-t)} d_q t$.

The polynomial $M_{n,q}^{-1,-1} f$ is well defined for any function f defined on $[0, 1]$, bounded in a neighborhood of 0 (condition $C(-1)$). It verifies $M_{n,q}^{-1,-1} f(0) = f(0)$ and $M_{n,q}^{-1,-1} f(0) = f(1)$, hence it preserves the affine functions.

Proposition 6 *If the function f is continuous on $[0, 1]$, then*

$$\lim_{\alpha \rightarrow -1} M_{n,q}^{\alpha,\alpha} f(x) = M_{n,q}^{-1,-1} f(x) \text{ for any } x \in [0, 1].$$

Proof. The q -binomial coefficients $b_{n,k,q}(x)$ are positive and form a partition of the unity. Hence it is sufficient to prove that $\lim_{\alpha \rightarrow -1} f_{n,k,q}^{\alpha,\alpha} = f_{n,k,q}^{-1,-1}$ for any k .

For $k = 1, \dots, n-1$, we compute $f_{n,k,q}^{\alpha,\alpha} - f_{n,k,q}^{-1,-1} = \frac{\int_0^1 t^{k+\alpha} (1-qt)_q^{n-k+\alpha} (f(q^{\alpha+1}t) - f(t)) d_q t}{B_q(k+\alpha+1, n-k+\alpha+1)} + \frac{\int_0^1 t^{k+\alpha} (1-qt)_q^{n-k+\alpha} f(t) d_q t}{B_q(k+\alpha+1, n-k+\alpha+1)} - \frac{\int_0^1 t^{k-1} (1-qt)_q^{n-k-1} f(t) d_q t}{B_q(k, n-k)} = F_1 + F_2 + F_3.$

We consider $I_k = \frac{\int_0^1 t^{k+\alpha} (1-qt)_q^{n-k+\alpha} (f(q^{\alpha+1}t) - \tilde{f}_k(t)) d_q t}{B_q(k+\alpha+1, n-k+\alpha+1)}$, with $\tilde{f}_0(t) = f(0)$, $\tilde{f}_n(t) = f(1)$ and $\tilde{f}_k(t) = f(t)$, $k = 1, \dots, n-1$ and prove that $\lim_{\alpha \rightarrow -1} I_k = 0$, for $k = 0, \dots, n$.

We use the additivity of the modulus of continuity of f , Beta integrals and we set

$\delta = [\alpha + 1] / [n + 2\alpha + 2]$. We have $|f(q^{\alpha+1}t) - f(0)| \leq \omega(f, t) \leq \omega(f, \delta)(1 + t/\delta)$,

hence $|I_0| \leq \omega(f, \delta)(1 + \frac{1}{\delta} \int_0^1 t^{\alpha+1} (1-qt)_q^{n+\alpha} d_q t / \int_0^1 t^\alpha (1-qt)_q^{n+\alpha} d_q t) \leq 2\omega(f, \frac{[\alpha+1]}{[n+2\alpha+2]}).$

For $k = n$, we have $|f(q^{\alpha+1}t) - f(1)| \leq \omega(f, 1 - q^{\alpha+1}t)$, hence

$|I_n| \leq \omega(f, \delta)(1 + \int_0^1 t^{\alpha+n} (1-qt)_q^{\alpha+1} d_q t / (\delta \int_0^1 t^{\alpha+n} (1-qt)_q^\alpha d_q t)) \leq 2\omega(f, \frac{[\alpha+1]}{[n+2\alpha+2]}).$

For $k = 1, \dots, n-1$, we have $|f(q^{\alpha+1}t) - f(t)| \leq \omega(f, 1 - q^{\alpha+1})$ and

$|I_k| \leq \omega(f, 1 - q^{\alpha+1})$. As $f_{n,0,q}^{\alpha,\alpha} - f(0) = I_0$ and $f_{n,n,q}^{\alpha,\alpha} - f(1) = I_n$, the result follows for $k = 0$ and n .

For the other cases, $F_1 = I_k$ vanishes when α tends to -1 . The upper term of F_2 is the q -integral $(1-q) \sum_{j=0}^n q^{j(k+\alpha+1)} (1-q^{j+1})^{n-k+\alpha} f(q^j)$. This serie is uniformly convergent, hence its limit when α tends to -1 is the upper term of F_3 . At last the lower term of F_2 tends to the lower term of A_3 because Γ_q is continuous. ■

Numerous properties shown in the case $\alpha, \beta > -1$ are still true if $\alpha = \beta = -1$. Some of them are given in the following.

Proposition 7 *If the function f is continuous at the points 0 and 1, and verifies the*

condition $C(-1)$, we have :

$$D_q M_{n,q}^{-1,-1} f(x) = M_{n-1,q}^{0,0} \left(D_q f \left(\frac{\cdot}{q} \right) \right) (qx), x \in [0, 1]. \quad (13)$$

Proof. The expressions for $(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta})$ in proposition (1) hold if $\alpha = \beta = -1$ and $k = 1, \dots, n-2$. For the two other terms we have

$$[n-1] (f_{n,1}^{-1,-1} - f(0)) = -[t^{n-1}(f(t) - f(1))]_0^1 + \int_0^1 (1-qt)_q^{n-1} D_q f(t) d_q t \text{ and}$$

$$[n-1] (f_{n,n-1}^{-1,-1} - f(1)) = -[(1-t)_q^{n-1}(f(t) - f(0))]_0^1 + \int_0^1 (qt)^{n-1} D_q f(t) d_q t.$$

The first terms vanish since f is continuous at 0 and 1. ■

Theorem 6

1. $M_{n,q}^{-1,-1} f(x) = \sum_{j=0}^{\infty} \Phi_{j,n,q}^{-1,-1}(x) f(q^j)$, where $\Phi_{j,n,q}^{-1,-1}$ is defined in formula (6) with $\alpha = \beta = -1$. The sequence $\Phi_{r,n,q}^{-1,-1}, \Phi_{r-1,n,q}^{-1,-1}, \dots, \Phi_{0,n,q}^{-1,-1}$ is totally positive. Consequently the operator $M_{n,q}^{-1,-1}$ diminishes the number of sign changes and preserves the monotony.

2. If f is continuous on $[0, 1]$, then $\|M_{n,q}^{-1,-1} f - f\|_{\infty} \leq C \omega \left(f, \frac{1}{\sqrt{[n]_q}} \right)$, (theorem 2).

3. If $\lim_{n \rightarrow \infty} q_n = 1$ and if the function f is bounded on $[0, 1]$, then

(a) $\lim_{n \rightarrow \infty} M_{n,q_n}^{-1,-1} f(x) = f(x)$ if f is continuous at the point $x \in]0, 1[$, (theorem 3).

(b) $\lim_{n \rightarrow \infty} [n]_{q_n} (M_{n,q_n}^{-1,-1} f(x) - f(x)) = f''(x)$ if the function f admits a second derivative at the point $x \in]0, 1[$, (theorem 4).

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